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Exact solutions for heat and mass transfer in a falling laminar film

J.L. Lockshin *, M.K. Zakharov

Department of Processes and Apparatuses of Chemical Engineering, Moscow State Academy of Fine Chemical Technology, Prospekt Vernadskogo, 86, Moscow, Russia 117571

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Abstract

Heat or mass transfer is considered in a falling laminar film with a parabolic speed profile. The speed profile is determined by gas flow at the film's free surface. Exact analytical solutions have been obtained for temperature or concentration profiles under general boundary conditions of the third kind, as well as for the corresponding Nusselt (Sherwood) numbers. The two cases of most practical importance are discussed in more detail: (I) heat or mass transfer with the wall and (II) mass transfer with the gas. Subsequent analysis of the solution has resulted in good approximate formulas for these two cases.

1. Introduction

Exchange devices employing falling films are used extensively in chemical engineering [1]. Principal advantages are high rates of heat or mass transfer and short contact time (a major consideration when dealing with heat-sensitive materials). Falling-film absorbers are used for dissolving gases in liquids, separation of gas mixtures and removal of unwanted components from a gas flow. Mass transfer between a wall and a liquid film takes place during dissolution, corrosion or anode dissolution of metals in electrochemical processes, etc. In these cases, information pertaining to the temperature or concentration profiles is important, as are the Nusselt or Sherwood numbers.

In this article, heat or mass transfer is considered in a falling laminar film with a generalized parabolic speed profile determined by the action of gas flow on the free surface of the film. Until now, exact solutions for heat or mass transfer in laminar liquid films have been given only for cases in which traction on the free surface of the film is considered negligible, and almost exclusively for boundary conditions of the first kind. Exact solutions are given for heat or mass transfer in a falling laminar film with a generalized parabolic speed profile under boundary conditions of the third kind. Analytically derived approximate expressions for calculating these solutions are given which can be used to greatly increase the ease and speed of calculations.

Additionally, since the solution is exact, comparison of the theoretical Nusselt (Sherwood) numbers with experimental data can help to determine how well the actual speed profile in a falling laminar film matches the theoretical parabolic distribution.

2. Model description

We will consider the film to be laminar. The speed profile of the falling film is considered to stabilize very quickly. If the speed of the liquid is taken to be zero at the wall, and traction at the gas interface to be equal to τ, then in a laminar film flowing down a vertical wall the local vertical speed v at distance y from the wall will equal

$$v(y) = -\frac{g}{2\nu}y^2 + \frac{g\delta(1-\Omega)}{\nu}y. \qquad (1)$$

Here g is the acceleration of gravity, ν is the cinematic viscosity of the liquid, $\Omega \equiv -(1/\rho\delta)\,\tau/g$ is a dimensionless parameter, expressing the relative strength of

* Corresponding author.
E-mail address: lock@dialup.ptt.ru (J.L. Lockshin).

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Nomenclature

a	temperature conductivity
C	concentration
D	diffusion coefficient
g	acceleration of gravity
T	temperature
v	speed
y, z	coordinates
Greek letters	
α	heat or mass transfer coefficient
δ	film thickness
λ	heat conductivity
ν	cinematic viscosity
ρ	density
τ	traction
Superscripts	
I	case I
II	case II
Subscripts	
av	integral flow average
g	gas interface
w	wall interface

traction at the film surface and mass forces, ρ is the density of the liquid. Formula (1) is easily found by direct integration of the Navier–Stokes equation if the viscosity terms dominate, pressure is considered constant and the speed of the liquid in other directions is equal to zero.

If the direction of the gas flow is the same as the direction of gravity, then $\Omega < 0$; if the gas is static, or the friction between the gas and the liquid can be considered negligible, then $\Omega = 0$; if the direction of the gas flow is directed against gravity, then $\Omega > 0$.

The speed profiles for various values of Ω are shown in Fig. 1. In this article we will consider only falling films, i.e. $\Omega \leqslant 0.5$. It should be pointed out that it is unlikely for a film to retain laminar flow at any significantly non-zero values of Ω, although the exact boundaries will be different in each concrete case. It has been shown [2] that in the case of transfer between the film and the wall, the effect of waves is negligible. Therefore, the model discussed here should be applicable to wavy-laminar flow for this situation. It will probably be inapplicable to wavy-laminar flow in the case of transfer between the film and the gas.

All physical properties of the liquid will be considered to be constant, and chemical reactions inside the film will be taken to be absent. Transfer by conduction (diffusion) will take place only along y, because transfer along z will be completely determined by the film flow. It has been shown [3] for $\Omega = 0$ that this is acceptable for Peclet numbers $Pe > 50$, which is almost always attained in practice. Heat transfer will then be defined by the equation

$$v(y)\frac{\partial T}{\partial z} = a\frac{\partial^2 T}{\partial y^2},$$

where z is the distance from the feed, a is the temperature conductivity and T denotes temperature.

We will suppose that the temperature at $z = 0$ will equal some initial value T_0:

$$T|_{z=0} = T_0.$$

Let α_w be the wall's transfer coefficient, which is considered to be the same for all z. Then the third-kind boundary condition at the wall can be written as

$$\lambda\frac{\partial T}{\partial y}\bigg|_{y=0} = \alpha_w\big(T_{y=0} - \varphi_w(z)\big),$$

where λ is the heat conductivity and $\varphi_w(z)$ is the temperature of the wall's surface.

In the same way, the boundary condition at the gas interface can be written as

$$\lambda\frac{\partial T}{\partial y}\bigg|_{y=\delta} = -\alpha_g\big(T_{y=\delta} - \varphi_g(z)\big).$$

The heat problem to be solved can be written as

$$\begin{aligned}
&\left(-\frac{g}{2\nu}y^2 + \frac{g(1-\Omega)\delta}{\nu}y\right)\frac{\partial T}{\partial z} = a\frac{\partial^2 T}{\partial y^2},\\
&\quad 0 < y < \delta,\ z > 0;\\
&T|_{z=0} = T_0, \quad 0 \leqslant y \leqslant \delta;\\
&\lambda\frac{\partial T}{\partial y}\bigg|_{y=0} = \alpha_w\big(T_{y=0} - \varphi_w(z)\big), \quad z > 0;\\
&\lambda\frac{\partial T}{\partial y}\bigg|_{y=\delta} = -\alpha_g\big(T_{y=\delta} - \varphi_g(z)\big), \quad z > 0.
\end{aligned} \tag{2}$$

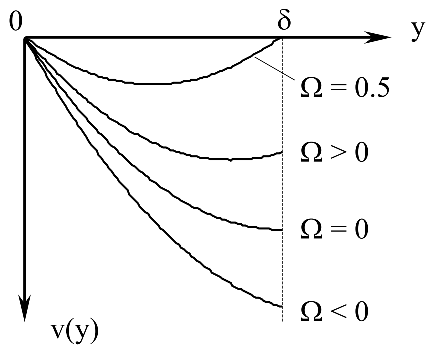


Fig. 1. Speed profiles in a falling laminar film.

Two major cases are of interest: (I) heat or mass transfer between the wall and the film, and (II) mass transfer between the gas and the film (absorption). For case I, heat exchange between the gas and the film can be considered to be negligible, while mass exchange will not take place; $\alpha_g = 0$. For case II, mass transfer between the film and the wall is absent, i.e. $\alpha_w = 0$. The superscripts I and II will denote case I and case II; if no superscript is present, then that formula pertains to both cases.

We will rewrite the problem (2) in dimensionless form. The dimensionless coordinates will be

$$\xi \equiv 1 - \frac{1}{\delta(1-\Omega)} y \quad \text{and} \quad \theta \equiv \frac{z}{\delta},$$

while the dimensionless temperature will equal

$$\Theta^{\mathrm{I}}(\xi,\theta) \equiv \frac{T(\xi,\theta) - T_0}{\varphi_w(0) - T_0} \quad \text{or} \quad \Theta^{\mathrm{II}}(\xi,\theta) \equiv \frac{T(\xi,\theta) - T_0}{\varphi_g(0) - T_0}.$$

Then the problem (2) can be written as

$$\begin{aligned}
&Pe(1-\Omega)^4(1-\xi^2)\frac{\partial\Theta}{\partial\theta} = \frac{\partial^2\Theta}{\partial\xi^2}, \quad \phi < \xi < 1, \ \theta > 0;\\
&\Theta|_{\theta=0} = 0, \quad \phi \leqslant \xi \leqslant 1;\\
&\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=1} = Bi_w(\vartheta_w(\theta) - \Theta_{\xi=1}), \quad \theta > 0;\\
&\left.\frac{\partial\Theta}{\partial\xi}\right|_{\xi=\phi} = -Bi_g(\vartheta_g(\theta) - \Theta_{\xi=\phi}), \quad \theta > 0.
\end{aligned} \tag{3}$$

Here

$$\phi = -\frac{\Omega}{1-\Omega}, \quad Pe = \frac{g\delta^3}{2va}$$

is the Peclet number of the liquid film,

$$Bi_w = \frac{\alpha_w}{\lambda}\delta(1-\Omega) \quad \text{and} \quad Bi_g = \frac{\alpha_g}{\lambda}\delta(1-\Omega)$$

are the Biot numbers at the wall and gas interfaces, respectively,

$$\vartheta_w^{\mathrm{I}}(\theta) = \frac{\varphi_w(\theta) - T_0}{\varphi_w(0) - T_0} \quad \text{and} \quad \vartheta_g^{\mathrm{I}}(\theta) = \frac{\varphi_g(\theta) - T_0}{\varphi_w(0) - T_0},$$

$$\vartheta_w^{\mathrm{II}}(\theta) = \frac{\varphi_w(\theta) - T_0}{\varphi_g(0) - T_0} \quad \text{and} \quad \vartheta_g^{\mathrm{II}}(\theta) = \frac{\varphi_g(\theta) - T_0}{\varphi_g(0) - T_0}.$$

Mass transfer in a falling mathematical film can be described by (3) as well, if a and λ are replaced by the diffusion coefficient D and T is replaced by the concentration C.

Problems analogous to (3) have been previously considered only for $\Omega = 0$ and almost exclusively for boundary conditions of the first kind ($Bi_w \to \infty$ or $Bi_g \to \infty$). The exact solution for these conditions for both cases in terms of infinite sums is given in [4] and [5]; the latter also gives asymptotic expressions for the sum terms. An exhaustive review of other results can be found in [6], all of which pertain to $\Omega = 0$.

For practical purposes, the intensity of heat or mass transfer (Nusselt or Sherwood number) and its integral average is of interest. The Nusselt (Sherwood) number is defined as the ratio between the amount of heat (mass) transferred to the film and the temperature (concentration) difference between the heat (mass) source and the average heat (concentration) of the film. In this article, the Nusselt (Sherwood) number will be found for the case when $\vartheta_w(\theta)$ and $\vartheta_g(\theta)$ are constant.

The average temperature (concentration) of the film at a given height will be calculated as the integral flow mean

$$\Theta_{av}(\theta) = \frac{\int_\phi^1 \Theta(\xi,\theta)v(\xi)\,d\xi}{\int_\phi^1 v(\xi)\,d\xi} = \frac{\int_\phi^1 \Theta(\xi,\theta)(1-\xi^2)\,d\xi}{\int_\phi^1 (1-\xi^2)\,d\xi}. \tag{4}$$

In dimensionless terms, the Nusselt (Sherwood) number will equal for case I

$$Nu^{\mathrm{I}}(\theta) = \frac{\left.\frac{\partial\Theta^{\mathrm{I}}}{\partial\xi}\right|_{\xi=1}}{1 - \Theta_{av}^{\mathrm{I}}}, \tag{5}$$

while for case II

$$Nu^{\mathrm{II}}(\theta) = -\frac{\left.\frac{\partial\Theta^{\mathrm{II}}}{\partial\xi}\right|_{\xi=\phi}}{1 - \Theta_{av}^{\mathrm{II}}}. \tag{6}$$

The average Nusselt number for both cases equals

$$Nu_{av}(\theta) = \frac{1}{\theta}\int_0^\theta Nu(x)\,dx. \tag{7}$$

3. Solution

3.1. General approach

Problem (3) can be most easily solved by use of Fourier series. Both sides of the differential equation and the initial condition will be multiplied by some twice-differentiable function $\Psi(\xi)$ and integrated by ξ from 0 to ϕ. Let

$$\int_\phi^1 (1-\xi^2)\Psi(\xi)\Theta(\xi,\theta)\,d\xi = \overline{\Theta}(\theta),$$

then after integrating the right-hand side of the differential equation in (3) twice by parts, we will find

$$Pe(1-\Omega)^4\frac{d\overline{\Theta}}{d\theta} = \Psi\left.\frac{\partial\Theta}{\partial\xi}\right|_\phi^1 - \Theta\left.\frac{d\Psi}{d\xi}\right|_\phi^1 + \int_\phi^1 \Theta\frac{d^2\Psi}{d\xi^2}\,d\xi. \tag{8}$$

The initial condition in (3) can be written as

$$\overline{\Theta}(0) = 0. \tag{9}$$

We will now require the function $\Psi(\xi)$ to satisfy the homogenous Sturm–Liouville problem

$$\begin{aligned} &\frac{\mathrm{d}^2\Psi}{\mathrm{d}\xi^2} = -\lambda^2\left(1-\xi^2\right)\Psi, \quad \phi < \xi < 1; \\ &\left.\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}\right|_{\xi=1} + Bi_{\mathrm{w}}\Psi|_{\xi=1} = 0; \\ &\left.\frac{\mathrm{d}\Psi}{\mathrm{d}\xi}\right|_{\xi=\phi} - Bi_{\mathrm{g}}\Psi|_{\xi=\phi} = 0. \end{aligned} \tag{10}$$

Using (10), Eq. (8) can be written as

$$\begin{aligned} Pe(1-\Omega)^4\frac{\mathrm{d}\overline{\Theta}}{\mathrm{d}\theta} = {} & -\lambda^2\overline{\Theta} + Bi_{\mathrm{g}}\Psi(\phi)\vartheta_{\mathrm{g}}(\theta) \\ & + Bi_{\mathrm{w}}\Psi(1)\vartheta_{\mathrm{w}}(\theta). \end{aligned} \tag{11}$$

The solution of the system (11), (9) is

$$\begin{aligned} \overline{\Theta}(\theta) = {} & \frac{1}{Pe(1-\Omega)^4}\int_0^\theta \left[Bi_{\mathrm{g}}\Psi(\phi)\vartheta_{\mathrm{g}}(x) + Bi_{\mathrm{w}}\Psi(1)\vartheta_{\mathrm{w}}(x)\right] \\ & \times \exp\left(-\frac{\lambda^2(\theta-x)}{Pe(1-\Omega)^4}\right)\mathrm{d}x. \end{aligned} \tag{12}$$

We will now determine $\Psi(\xi)$ on the basis of (10). The general solution of the differential equation has the form

$$\begin{aligned} \Psi(\xi) = {} & \left[A_1\Phi\left(\frac{1-\lambda}{4},\frac{1}{2};\lambda\xi^2\right)\right. \\ & \left. + A_2\xi\Phi\left(\frac{3-\lambda}{4},\frac{3}{2};\lambda\xi^2\right)\right]\exp\left(-\frac{\lambda}{2}\xi^2\right), \end{aligned} \tag{13}$$

where A_1 and A_2 are unknown constants and $\Phi(a,b;x)$ is the confluent hypergeometric function:

$$\Phi(a,b;x) = 1 + \sum_{n=1}^{\infty}\frac{\Gamma(a+n)}{\Gamma(a)}\frac{\Gamma(b)}{\Gamma(b+n)}\frac{x^n}{n!}.$$

For brevity, let

$$\begin{aligned} f(\xi) &= \Phi\left(\frac{1-\lambda}{4},\frac{1}{2};\lambda\xi^2\right)\exp\left(-\frac{\lambda}{2}\xi^2\right), \\ g(\xi) &= \xi\Phi\left(\frac{3-\lambda}{4},\frac{3}{2};\lambda\xi^2\right)\exp\left(-\frac{\lambda}{2}\xi^2\right). \end{aligned}$$

Then the boundary conditions in (10) can be written as

$$\begin{aligned} &A_1\left(\left.\frac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{\xi=1} + Bi_{\mathrm{w}}f(1)\right) + A_2\left(\left.\frac{\mathrm{d}g}{\mathrm{d}\xi}\right|_{\xi=1} + Bi_{\mathrm{w}}g(1)\right) = 0; \\ &A_1\left(\left.\frac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{\xi=\phi} - Bi_{\mathrm{g}}f(\phi)\right) + A_2\left(\left.\frac{\mathrm{d}g}{\mathrm{d}\xi}\right|_{\xi=\phi} - Bi_{\mathrm{g}}g(\phi)\right) = 0. \end{aligned} \tag{14}$$

In order for nontrivial solutions of this system to exist, it is necessary and sufficient for the determinant of (14) to be equal to zero, i.e.

$$\begin{aligned} &\left(\left.\frac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{\xi=\phi} - Bi_{\mathrm{g}}f(\phi)\right)\left(\left.\frac{\mathrm{d}g}{\mathrm{d}\xi}\right|_{\xi=1} + Bi_{\mathrm{w}}g(1)\right) \\ &\quad - \left(\left.\frac{\mathrm{d}g}{\mathrm{d}\xi}\right|_{\xi=\phi} - Bi_{\mathrm{g}}g(\phi)\right)\left(\left.\frac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{\xi=1} + Bi_{\mathrm{w}}f(1)\right) = 0. \end{aligned} \tag{15}$$

Those values of λ that satisfy this equation are the eigenvalues of the system (10) and will be denoted λ_n. The corresponding eigenfunctions will be denoted $\Psi_n(\xi)$. For brevity, we will write (15) in the form $\Delta(\lambda,\phi,Bi_{\mathrm{g}},Bi_{\mathrm{w}}) = 0$.

The constants A_1 and A_2 themselves can assume infinitely many values. However, it is convenient to determine these constants on the basis of the second-kind boundary condition in (10):

$$\begin{aligned} A_1 &= \left.\frac{\mathrm{d}g}{\mathrm{d}\xi}\right|_{\xi=\xi_1}, \\ A_2 &= -\left.\frac{\mathrm{d}f}{\mathrm{d}\xi}\right|_{\xi=\xi_1}, \end{aligned} \tag{16}$$

where $\xi_1 = \phi$ for case I and $\xi_1 = 1$ for case II.

The first eigenvalues for various values of Bi_{w} and Bi_{g} as functions of ϕ are shown in Fig. 2.

It is now possible to write out the solution to the initial system (3). In terms of Fourier series, the solution can be written in the form

$$\Theta(\xi,\theta) = \sum_{n=1}^{\infty}B_n(\theta)\Psi_n(\xi), \tag{17}$$

where the function $\Psi_n(\xi)$ is given by Eqs. (13) and (16), and corresponds to the eigenvalue λ_n. If both sides of (17) are multiplied by $(1-\xi^2)\Psi_n(\xi)$ and integrated by ξ from ϕ to 1, then the left-hand side will equal $\overline{\Theta}_n(\theta)$. The right-hand side will simplify to $B_n(\theta)\int_\phi^1(1-\xi^2)\Psi_n^2\,\mathrm{d}\xi$, because in view of the properties of $\Psi_n(\xi)$

$$\begin{aligned} \int_\phi^1\left(1-\xi^2\right)\Psi_n\Psi_m\,\mathrm{d}\xi &= -\frac{1}{\lambda_n^2}\int_\phi^1\frac{\mathrm{d}^2\Psi_n}{\mathrm{d}\xi^2}\Psi_m\,\mathrm{d}\xi \\ &= -\frac{1}{\lambda_n^2}\left[\left.\frac{\mathrm{d}\Psi_n}{\mathrm{d}\xi}\Psi_m\right|_\phi^1 - \left.\Psi_n\frac{\mathrm{d}\Psi_m}{\mathrm{d}\xi}\right|_\phi^1 \right. \\ &\qquad \left. + \int_\phi^1\frac{\mathrm{d}^2\Psi_n}{\mathrm{d}\xi^2}\Psi_m\,\mathrm{d}\xi\right] \\ &= \frac{\lambda_m^2}{\lambda_n^2}\int_\phi^1\left(1-\xi^2\right)\Psi_n\Psi_m\,\mathrm{d}\xi, \end{aligned}$$

so that if $n \neq m$ $\int_\phi^1(1-\xi^2)\Psi_n\Psi_m\,\mathrm{d}\xi = 0$. (The system of functions $\{\Psi_n(\xi)\}$ is orthogonal.)

Therefore, we find

$$B_n(\theta) = \frac{\overline{\Theta}_n(\xi,\theta)}{\int_\phi^1\left(1-\xi^2\right)\Psi_n^2(\xi)\,\mathrm{d}\xi}.$$

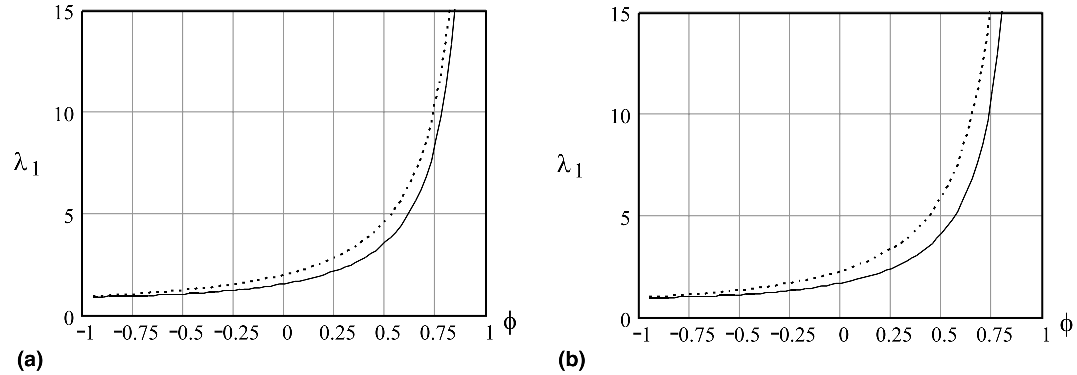


Fig. 2. The first eigenvalue for (a) $Bi_w = 10$ (case I), $Bi_g = 10$ (case II) and (b) $Bi_w = \infty$ (case I), $Bi_g = \infty$ (case II). Solid lines are used for case I, dotted lines are used for case II.

Putting this equation into (17) and using (12) gives the solution

$$\Theta(\xi,\theta) = \frac{1}{Pe(1-\Omega)^4}\sum_{n=1}^{\infty}\left[\left\{\Psi_n(\xi)\int_0^{\theta}\left[Bi_g\Psi_n(\phi)\vartheta_g(x) + Bi_w\Psi_n(1)\vartheta_w(x)\right]\exp\left(-\frac{\lambda_n^2(\theta-x)}{Pe(1-\Omega)^4}\right)\mathrm{d}x\right\} \Bigg/ \left\{\int_{\phi}^{1}(1-\xi^2)\Psi_n^2\,\mathrm{d}\xi\right\}\right]. \tag{18}$$

If $\vartheta_g(x)$ and $\vartheta_w(x)$ are constant (and therefore equal to 1), then

$$\Theta(\xi,\theta) = 1 - \sum_{n=1}^{\infty}\frac{\left[Bi_g\Psi_n(\phi) + Bi_w\Psi_n(1)\right]}{\lambda_n^2\int_{\phi}^{1}\left(1-\xi^2\right)\Psi_n^2\,\mathrm{d}\xi}\Psi_n(\xi) \times \exp\left(-\frac{\lambda_n^2}{Pe(1-\Omega)^4}\theta\right). \tag{19}$$

It can be seen from the differential equation in (10) that

$$\int_{\phi}^{1}\left(1-\xi^2\right)\Psi_n(\xi)\,\mathrm{d}\xi = -\frac{1}{\lambda_n^2}\left.\frac{\mathrm{d}\Psi_n}{\mathrm{d}\xi}\right|_{\phi}^{1},$$

so that the average temperature (concentration) given by (4) can be easily found:

$$\Theta_{av}(\theta) = 1 + \frac{3}{2-3\phi+\phi^3} \times \sum_{n=1}^{\infty}\frac{\left[Bi_g\Psi_n(\phi) + Bi_w\Psi_n(1)\right]}{\lambda_n^4\int_{\phi}^{1}\left(1-\xi^2\right)\Psi_n^2\,\mathrm{d}\xi}\left.\frac{\mathrm{d}\Psi_n}{\mathrm{d}\xi}\right|_{\phi}^{1} \times \exp\left(-\frac{\lambda_n^2}{Pe(1-\Omega)^4}\theta\right). \tag{20}$$

Using the boundary conditions in (10) to calculate

$$\left.\frac{\mathrm{d}\Psi_n}{\mathrm{d}\xi}\right|_{\phi}^{1}$$

and inserting (20) into the formulas (5) and (6), we find

$$Nu(\theta) = \frac{\left(2-3\phi+\phi^3\right)}{3}\left[\left\{\sum_{n=1}^{\infty}\frac{\Psi_n^2(\xi_1)}{\lambda_n^2\int_{\phi}^{1}\left(1-\xi^2\right)\Psi_n^2(\xi)\,\mathrm{d}\xi} \times \exp\left(-\frac{\lambda_n^2}{Pe(1-\Omega)^4}\theta\right)\right\} \Bigg/ \left\{\sum_{n=1}^{\infty}\frac{\Psi_n^2(\xi_1)}{\lambda_n^4\int_{\phi}^{1}\left(1-\xi^2\right)\Psi_n^2(\xi_1)\,\mathrm{d}\xi} \times \exp\left(-\frac{\lambda_n^2}{Pe(1-\Omega)^4}\theta\right)\right\}\right], \tag{21}$$

where $\xi_1 = 1$, $Bi_g = 0$, $\vartheta_w \equiv 1$ for case I and $\xi_1 = \phi$, $Bi_w = 0$, $\vartheta_g \equiv 1$ for case II. It can be seen that for either case as $\theta \to \infty$

$$Nu(\theta) \to \frac{2-3\phi+\phi^3}{3}\lambda_1^2. \tag{22}$$

If $\phi = 0$, then for case I the first root of (15) equals $\lambda_1^{\mathrm{I}} = 1.6815$ and $\lim_{\theta\to\infty} Nu^{\mathrm{I}}(\theta) = 1.8850$; this is Nusselt's classic result. For case II $\lambda_1^{\mathrm{II}} = 2.2628$ and $\lim_{\theta\to\infty} Nu^{\mathrm{II}}(\theta) = 3.4135$. The behavior of the Nusselt numbers as functions of ϕ will be discussed in more detail later.

Eq. (21) can be rewritten in the form

$$Nu(\theta) = \frac{2-3\phi+\phi^3}{3}Pe(1-\Omega)^4 \times \frac{\mathrm{d}}{\mathrm{d}\theta}\left[-\ln\left(\sum_{n=1}^{\infty}\frac{\Psi_n^2(\xi_1)}{\lambda_n^2\int_{\phi}^{1}\left(1-\xi^2\right)\Psi_n^2(\xi)\,\mathrm{d}\xi} \times \exp\left(-\frac{\lambda_n^2}{Pe(1-\Omega)^4}\theta\right)\right)\right].$$

Therefore, the average Nusselt number according to (7) will equal

$$Nu_{\mathrm{av}}(\theta) = \frac{2-3\phi+\phi^3}{3}\frac{Pe(1-\Omega)^4}{\theta} \times \ln\left(\sum_{n=1}^{\infty}\frac{\Psi_n^2(\xi_1)}{\lambda_n^2\int_\phi^1(1-\xi^2)\Psi_n^2(\xi_1)\,\mathrm{d}\xi} \times \exp\left(-\frac{\lambda_n^2}{Pe(1-\Omega)^4}\theta\right)\right). \tag{23}$$

3.2. Approximate expressions

We will now construct approximations to the exact solution (18). In this regard, it is necessary to find approximate expressions for $\Psi(\xi)$ from Eq. (16) and for the eigenfunctions λ_n from Eq. (15), as well as for the integrals $\int_\phi^1(1-\xi^2)\Psi_n^2(\xi)\mathrm{d}\xi$.

We will first consider Eq. (15). The function

$$f(\xi) = \Phi\left(\frac{1-\lambda}{4},\frac{1}{2};\lambda\xi^2\right)\exp\left(-\frac{\lambda}{2}\xi^2\right)$$

will be differentiated directly by use of the formula [7]

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi(a,b;x) = \frac{a}{b}\Phi(a+1,b+1;x),$$

while the derivative of

$$g(\xi) = \xi\Phi\left(\frac{3-\lambda}{4},\frac{3}{2};\lambda\xi^2\right)\exp\left(-\frac{\lambda}{2}\xi^2\right)$$

will be found by use of the formula [7]

$$\frac{\mathrm{d}}{\mathrm{d}x}\Phi(a,b;x) = \frac{b-1}{x}[\Phi(a,b;x)-\Phi(a,b-1;x)].$$

Then

$$\frac{\mathrm{d}f}{\mathrm{d}\xi} = \left[-\lambda\xi\Phi\left(\frac{1-\lambda}{4},\frac{1}{2};\lambda\xi^2\right) + \lambda(1-\lambda)\xi\Phi\left(\frac{5-\lambda}{4},\frac{3}{2};\lambda\xi^2\right)\right]\exp\left(-\frac{\lambda}{2}\xi^2\right)$$

and

$$\frac{\mathrm{d}g}{\mathrm{d}\xi} = \left[-\lambda\xi^2\Phi\left(\frac{3-\lambda}{4},\frac{3}{2};\lambda\xi^2\right) + \Phi\left(\frac{3-\lambda}{4},\frac{1}{2};\lambda\xi^2\right)\right]\exp\left(-\frac{\lambda}{2}\xi^2\right).$$

Asymptotic expansions for all relevant confluent hypergeometric functions are given in Appendix A. For case I, approximate expressions for the coefficients A_1 and A_2 from Eq. (16) can be found by using (A.3), (A.4) and (A.7), (A.8):

$$\Psi^{\mathrm{I}}(\xi) \cong \left[\cos\left(\frac{\lambda}{4}\zeta(\phi)\right)f(\xi) + \lambda\,\mathrm{sgn}(\phi)\sin\left(\frac{\lambda}{4}\zeta(\phi)\right)g(\xi)\right] \times (1-\phi^2)^{1/4}\exp\left(\frac{\lambda\phi^2}{2}\right), \tag{24}$$

where $\zeta(\phi) = 2|\phi|\sqrt{1-\phi^2} - 2\arccos(|\phi|) + \pi$.

Putting (A.3)–(A.8) into (15) and simplifying gives

$$\left(Bi_{\mathrm{w}} + 2^{4/3}3^{1/3}\frac{\Gamma(2/3)}{\Gamma(1/3)}\left(\lambda^{\mathrm{I}}\right)^{2/3}\right) \times \tan\left(\frac{\pi}{12} + (\mathrm{sgn}(\phi)\zeta(\phi)-\pi)\frac{\lambda^{\mathrm{I}}}{4}\right) + \sqrt{3}Bi_{\mathrm{w}} = 0. \tag{25}$$

For case II, use of (A.5), (A.6) and (A.9), (A.10) gives

$$\Psi^{\mathrm{II}}(\xi) \cong \frac{3^{-1/3}}{\Gamma(1/3)}2^{7/6}\sqrt{\pi}\lambda^{-1/6} \times \left[\cos\left(\frac{\pi}{12}-\frac{\lambda}{4}\pi\right)f(\xi) - \lambda\sin\left(\frac{\pi}{12}-\frac{\lambda}{4}\pi\right)g(\xi)\right]. \tag{26}$$

Putting (A.3)–(A.6), (A.9) and (A.10) into (15) and simplifying gives

$$\sqrt{1-\phi^2}\lambda^{\mathrm{II}}\tan\left(\frac{\pi}{12} + (\mathrm{sgn}(\phi)\zeta(\phi)-\pi)\frac{\lambda^{\mathrm{II}}}{4}\right) + Bi_{\mathrm{g}} = 0. \tag{27}$$

Numerical comparison of the first and second roots of Eqs. (25) and (27) and of Eq. (15) is shown in Fig. 3. The error for other eigenvalues can be only less. For practical applications, we suggest that the first eigenvalue be calculated exactly from (15).

For boundary conditions of the first kind ($Bi_{\mathrm{w}} \to \infty$ or $Bi_{\mathrm{g}} \to \infty$), Eqs. (25) and (27) become

$$\lambda_n^{\mathrm{I}} = \frac{(5\pi/3) + 4\pi(n-1)}{2\arccos(\phi) - 2\phi\sqrt{1-\phi^2}}$$

for case I and

$$\lambda_n^{\mathrm{II}} = \frac{(7\pi/3) + 4\pi(n-1)}{2\arccos(\phi) - 2\phi\sqrt{1-\phi^2}}$$

for case II.

If we now put $\phi = 0$ (negligible friction between the film's free surface and the gas), we will get the known [5] asymptotic formulas

$$\lambda_n^{\mathrm{I}} \cong \frac{5}{3} + 4(n-1)$$

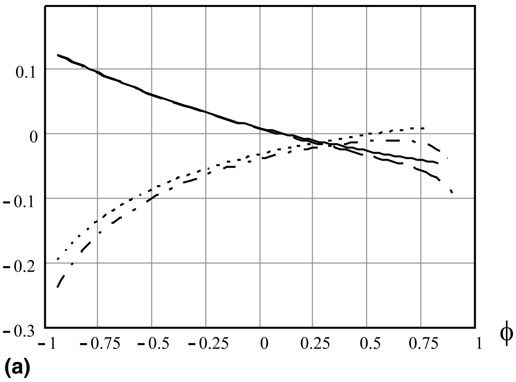
for case I and

$$\lambda_n^{\mathrm{II}} \cong \frac{7}{3} + 4(n-1)$$

for case II.

We will now find approximate expressions for the integrals

$$\int_\phi^1(1-\xi^2)\Psi_n^2(\xi)\,\mathrm{d}\xi.$$



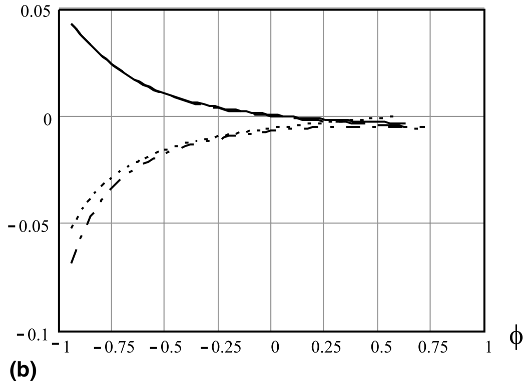


Fig. 3. Relative error between the exact value of the first (a) and second (b) eigenvalues and the first and second roots of (25) for case I and of (27) for case II.

It can be seen from Eq. (B.1) that

$$\int_\phi^1 (1-\xi^2)\Psi_n^2\,\mathrm{d}\xi = \frac{1}{2\lambda_n}\left(\frac{\partial\Psi_n}{\partial\xi}\frac{\partial\Psi_n}{\partial\lambda_n} - \Psi_n\frac{\partial^2\Psi_n}{\partial\lambda_n\partial\xi}\right)_\phi^1,$$

which can be rewritten using the boundary equations in (10)

$$\int_\phi^1 (1-\xi^2)\Psi_n^2\,\mathrm{d}\xi = -\frac{1}{2\lambda_n}\Psi_n(1)\frac{\partial}{\partial\lambda_n}\left(\frac{\partial\Psi_n}{\partial\xi} + Bi_{\mathrm{w}}\Psi_n\right)\bigg|_1 + \frac{1}{2\lambda_n}\Psi_n(\phi)\frac{\partial}{\partial\lambda_n}\left(\frac{\partial\Psi_n}{\partial\xi} - Bi_{\mathrm{g}}\Psi_n\right)\bigg|_\phi.$$

For case I, we will insert the constants A_1^{I} and A_2^{I} from (16): it can be seen that

$$\frac{\mathrm{d}\Psi^{\mathrm{I}}}{\mathrm{d}\xi}\bigg|_{\xi=\phi} \equiv 0$$

and

$$\frac{\partial}{\partial\lambda}\left(\frac{\mathrm{d}\Psi^{\mathrm{I}}}{\mathrm{d}\xi}\bigg|_{\xi=\phi}\right) \equiv 0.$$

Then

$$\int_\phi^1 (1-\xi^2)\left(\Psi_n^{\mathrm{I}}(\xi)\right)^2\mathrm{d}\xi = \frac{-1}{2\lambda_n^{\mathrm{I}}}\Psi_n^{\mathrm{I}}(1)\frac{\partial}{\partial\lambda_n^{\mathrm{I}}}\left(\frac{\partial\Psi_n^{\mathrm{I}}}{\partial\xi} - Bi_{\mathrm{w}}\Psi_n^{\mathrm{I}}\right)\bigg|_1 = -\frac{\Psi_n^{\mathrm{I}}(1)}{2\lambda_n^{\mathrm{I}}}\frac{\mathrm{d}}{\mathrm{d}\lambda_n^{\mathrm{I}}}\Delta\left(\phi,\lambda_n^{\mathrm{I}},0,Bi_{\mathrm{w}}\right).$$

However, because λ_n^{I} is the nth root of Eq. (15) when $Bi_{\mathrm{g}}=0$, we find from (25) that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_n^{\mathrm{I}}}\Delta\left(\lambda_n^{\mathrm{I}},\phi,0,Bi_{\mathrm{w}}\right) \cong \sqrt{\pi}2^{5/6}\left(1-\phi^2\right)^{1/4}\left(\lambda_n^{\mathrm{I}}\right)^{1/6} \times\left[\left(\frac{2}{3}\right)^{4/3}\frac{(\lambda_n^{\mathrm{I}})^{-1/3}}{\Gamma(1/3)}\sin\left(\frac{\pi}{12}+\frac{\lambda_n^{\mathrm{I}}}{4}(\mathrm{sgn}(\phi)\zeta(\phi)-\pi)\right) -(-1)^n\frac{\mathrm{sgn}(\phi)\zeta(\phi)-\pi}{4} \times\sqrt{\left(\frac{Bi_{\mathrm{w}}}{2}\frac{3^{-2/3}}{\Gamma(2/3)}+2^{1/3}(\lambda_n^{\mathrm{I}})^{2/3}\frac{3^{-1/3}}{\Gamma(1/3)}\right)^2+\left(Bi_{\mathrm{w}}\frac{3^{-2/3}}{\Gamma(2/3)}\frac{\sqrt{3}}{2}\right)^2}\,\right].$$

Therefore,

$$\Theta^{\mathrm{I}}(\xi,\theta) = \frac{1}{Pe(1-\Omega)^4}\sum_{n=1}^{\infty}C_n^{\mathrm{I}}\Psi_n^{\mathrm{I}}(\xi) \times\int_0^\theta \vartheta_{\mathrm{w}}(x)\exp\left(-\frac{\left(\lambda_n^{\mathrm{I}}\right)^2}{Pe(1-\Omega)^4}(\theta-x)\right)\mathrm{d}x, \tag{28}$$

where

$$C_n^{\mathrm{I}} = \left\{-14.27Bi_{\mathrm{w}}\left(1-\phi^2\right)^{-1/4}\left(\lambda_n^{\mathrm{I}}\right)^{5/6}\right\} \Big/ \left\{5\left(\lambda_n^{\mathrm{I}}\right)^{-1/3}\sin\left((\pi/12)+(\lambda_n^{\mathrm{I}}/4)(\mathrm{sgn}(\phi)\zeta(\phi)-\pi)\right) -(-1)^n(\mathrm{sgn}(\phi)\zeta(\phi)-\pi) \times\sqrt{\left(Bi_{\mathrm{w}}+1.837\left(\lambda_n^{\mathrm{I}}\right)^{2/3}\right)^2+3Bi_{\mathrm{w}}^2}\right\}, \tag{29}$$

and (24) can be used to calculate $\Psi_n^{\mathrm{I}}(\xi)$. In the same way, for case II expressions (16) and (27) can be used to find

$$\Theta^{\mathrm{II}}(\xi,\theta) = \frac{1}{Pe(1-\Omega)^4}\sum_{n=1}^{\infty}C_n^{\mathrm{II}}\Psi_n^{\mathrm{II}}(\xi) \times\int_0^\theta \vartheta_{\mathrm{g}}(x)\exp\left(-\frac{\left(\lambda_n^{\mathrm{II}}\right)^2}{Pe(1-\Omega)^4}(\theta-x)\right)\mathrm{d}x, \tag{30}$$

where

$$C_n^{\mathrm{II}} = -7.768 Bi_{\mathrm{g}} \left(\lambda_n^{\mathrm{II}}\right)^{7/6} \Big/ \Big\{ 4\sqrt{1-\phi^2} \sin\left((\pi/12) + (\mathrm{sgn}(\phi)\zeta(\phi) - \pi)(\lambda_n^{\mathrm{II}}/4)\right) - (-1)^n (\mathrm{sgn}(\phi)\zeta(\phi) - \pi) \sqrt{Bi_{\mathrm{g}}^2 + \left(\lambda_n^{\mathrm{II}}\right)^2 (1-\phi^2)} \Big\} \quad (31)$$

and $\Psi_n^{\mathrm{II}}(\xi)$ can be calculated by using (26).

Numerical results show that it is preferable to calculate the coefficients C_1^{I} and C_1^{II} exactly from (18). The relative error between the exact values of C_2^{I}, C_2^{II} and the values given by (29) and (31) is shown in Fig. 4; for higher-order coefficients, this error can be only less.

If $\vartheta_{\mathrm{w}}(x)$ and $\vartheta_{\mathrm{g}}(x)$ are constant, then for both cases (19) can be written as

$$\Theta(\xi,\theta) = 1 + \sum_{n=1}^{\infty} C_n \Psi_n(\xi) \exp\left(-\frac{\lambda_n^2}{Pe(1-\Omega)^4}\theta\right).$$

For boundary conditions of the first kind ($Bi_{\mathrm{w}} \to \infty$ or $Bi_{\mathrm{g}} \to \infty$), Eqs. (29) and (31) can be simplified:

$$\Theta^{\mathrm{I}}(\xi,\theta) = 1 - 7.135 \sum_{n=1}^{\infty} \left[\left\{ (-1)^n \left(1-\phi^2\right)^{-1/4} \left(\lambda_n^{\mathrm{I}}\right)^{-7/6} \Psi_n^{\mathrm{I}}(\xi) \times \exp\left(-\frac{\left(\lambda_n^{\mathrm{I}}\right)^2}{Pe(1-\Omega)^4}\theta\right) \right\} \Big/ \{\mathrm{sgn}(\phi)\zeta(\phi) - \pi\} \right] \quad (32)$$

and

$$\Theta^{\mathrm{II}}(\xi,\theta) = 1 - 7.768 \sum_{n=1}^{\infty} \left[\left\{ (-1)^n \left(\lambda_n^{\mathrm{II}}\right)^{-5/6} \Psi_n^{\mathrm{II}}(\xi) \times \exp\left(-\left\{\frac{\left(\lambda_n^{\mathrm{II}}\right)^2}{Pe(1-\Omega)^4}\right\}\theta\right) \right\} \Big/ \{\mathrm{sgn}(\phi)\zeta(\phi) - \pi\} \right], \quad (33)$$

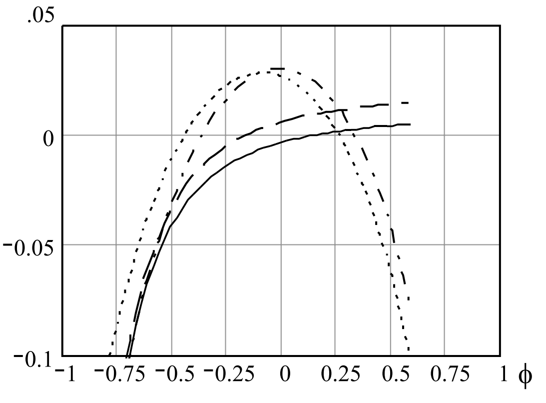


Fig. 4. Relative error between the exact value of the second coefficient C_2 in (18) and the approximate values calculated from (29) for case I and from (31) for case II.

respectively. These equations coincide with the numerical and asymptotic results given in [5] for $\phi = 0$.

Temperature profiles calculated from (32) and (33) are shown in Fig. 5(a) and (b) for $\phi = -0.2$ and Fig. 5(c) and (d) for $\phi = 0.2$ at dimensionless height

$$\frac{\theta}{Pe(1-\Omega)^4} = 0.1.$$

The Nusselt (Sherwood) number in accordance with (21) equals

$$Nu(\theta) = \frac{\left(2 - 3\phi + \phi^3\right)}{3} \left[\left\{ \sum_{n=1}^{\infty} C_n \Psi_n^2(\xi_1) \times \exp\left(-\frac{\lambda_n^2}{Pe(1-\Omega)^4}\theta\right) \right\} \Big/ \left\{ \sum_{n=1}^{\infty} \frac{C_n \Psi_n^2(\xi_1)}{\lambda_n^2} \times \exp\left(-\frac{\lambda_n^2}{Pe(1-\Omega)^4}\theta\right) \right\} \right]$$

while the average Nusselt (Sherwood) number in accordance with (23) equals

$$Nu_{\mathrm{av}}(\theta) = \frac{2 - 3\phi + \phi^3}{3} \frac{Pe(1-\Omega)^4}{\theta} \ln\left(\sum_{n=1}^{\infty} \frac{C_n \Psi_n^2(\xi_1)}{\lambda_n^2} \times \exp\left(-\frac{\lambda_n^2}{Pe(1-\Omega)^4}\theta\right) \right).$$

As before, $\xi_1 = 1$ for case I and $\xi_1 = \phi$ for case II. Graphs of $Nu(\theta)$ are shown in Fig. 6 for various values of ϕ, Bi_{w} and Bi_{g}.

4. Analysis

We will compare transfer processes for two cases: $\phi = -0.2$ ($\Omega = -0.25$) and $\phi = 0.2$ ($\Omega = 0.17$). Other values could have been chosen; however, the qualitative differences for $\phi < 0$ and $\phi > 0$ can be discussed, and these differences will become more (less) pronounced for values of ϕ that are further (closer) to 0. The reader should be reminded, however, that all conclusions are valid only for laminar films.

First, we will analyze the temperature profiles shown on Fig. 5. The transfer processes are more efficient for $\phi = 0.2$ in both case I and case II. This can be explained by considering the differences in the speed profile. The maximum local speed in the film for $\phi = -0.2$ is attained at an inner point. During the transfer process, the heat or mass flow reaches this layer; since it is faster than its neighboring layers, heat or mass transfer to the rest of the film will take place further down the wall, thereby decreasing the effectiveness of the transfer process. For $\phi = 0.2$, on the other hand, the maximum local speed is

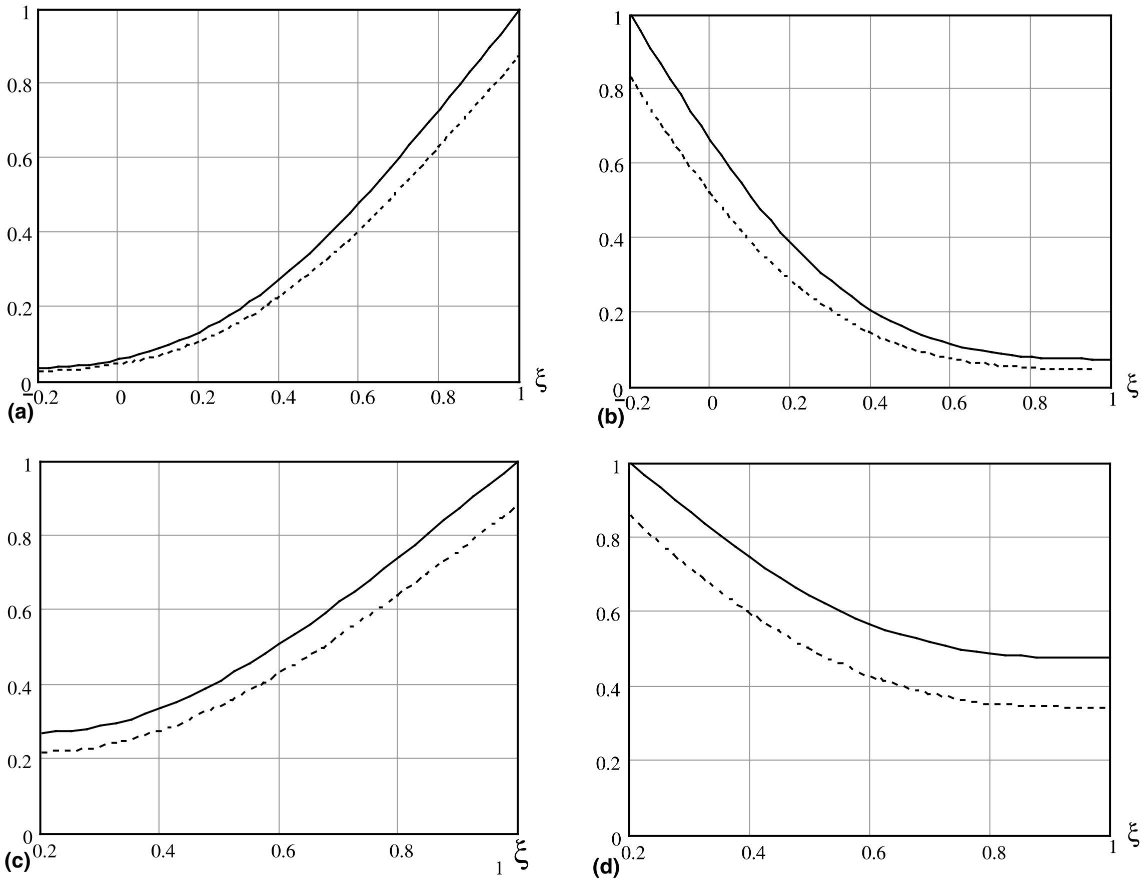


Fig. 5. Temperature profiles calculated at dimensionless height $\theta/Pe(1-\Omega)^4 = 0.1$ for $Bi = \infty$ (solid lines) and $Bi = 10$ (dotted lines). (a) case I: $\phi = -0.2,\ Bi_w = \infty$ and $Bi_w = 10$; (b) case II: $\phi = -0.2,\ Bi_g = \infty$ and $Bi_g = 10$; (c) case I: $\phi = 0.2,\ Bi_w = \infty$ and $Bi_w = 10$; (d) case II: $\phi = 0.2,\ Bi_g = \infty$ and $Bi_g = 10$.

attained at the liquid/gas interface and the transfer process is more efficient.

It can also be seen that in terms of mean temperature, transfer between the film and the gas is always more effective than transfer between the film and the wall for equal values of Bi_w and Bi_g. This is to be expected, as the average temperature is defined by (4), in which the integral weight function $(1-\xi^2)$ is minimal for the layers close to the wall and maximal for the layers close to the gas interface.

These conclusions are supported as well by the behavior of the Nusselt numbers shown in Fig. 6. First of all, the value of $Nu(\theta)$ is always greater for $\phi = 0.2$ than for $\phi = -0.2$. Additionally, the Nusselt numbers stabilize much faster for $\phi = 0.2$, indicating that the transfer process approaches completion at smaller values of θ. Secondly, the Nusselt numbers are always greater for case II for equal values of Bi_w and Bi_g . The heightened effectiveness of transfer for case II as compared with case I is also evident in the weaker influence of the Biot numbers: the difference between $Nu^{II}(\theta)$ for $Bi_g = 10$ and $Nu^{II}(\theta)$ $Bi_g = \infty$ is smaller than the corresponding difference between $Nu^{I}(\theta)$ for $Bi_w = 10$ and $Nu^{I}(\theta)$ for $Bi_w = \infty$.

Fig. 7 shows that for both cases the maximum Nusselt number is achieved at $\phi = 1$ for all values of Bi_w and Bi_g. This corresponds to the behavior of the eigenvalues as functions of ϕ shown in Fig. 2 in accordance with (22). Heat or mass transfer will take place more effectively for $\phi \to 1$ $(\Omega \to +\infty)$ because the fast layers at the gas interface create a heightened temperature or concentration difference through the film, leading to a more effective transfer process (see Fig. 1). If $\phi \to -1$ $(\Omega \to 0.5)$, then, first of all, the average speed of the film decreases. Secondly, the location of the maximum local speed inside the film, rather than on the gas interface, decreases the effectiveness of the transfer process as described earlier. Therefore, the gas flow should be concurrent to the film in order to intensify heat or mass transfer.

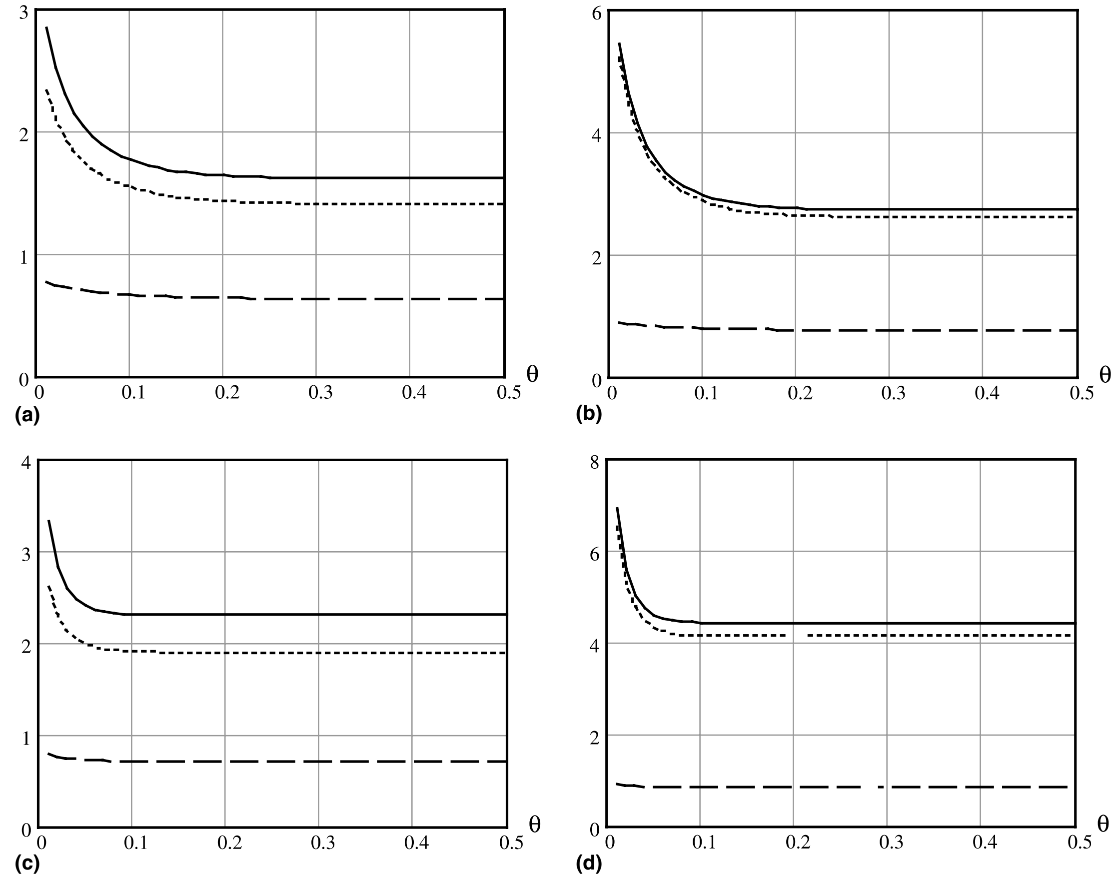


Fig. 6. Nusselt (Sherwood) numbers $Nu(\theta)$ for $Bi = \infty$ (solid lines), $Bi = 10$ (dotted lines) and $Bi = 1$ (dashed lines). (a) case I: $\phi = -0.2$, $Bi_w = 1$ and ∞; (b) case II: $\phi = -0.2$, $Bi_g = 1$, 10 and ∞; (c) case I: $\phi = 0.2$, $Bi_w = 1$, 10 and ∞; (d) case II: $\phi = 0.2$, $Bi_g = 1$, 10 and ∞.

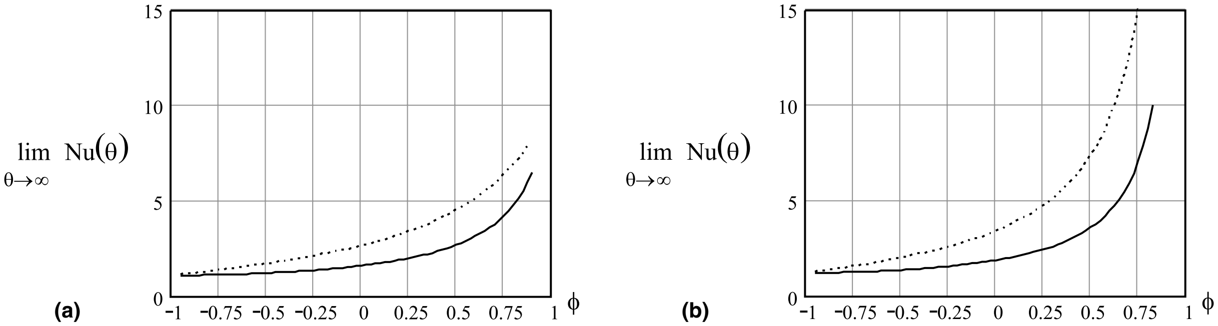


Fig. 7. The value of $\lim_{\theta\to\infty} Nu(\theta)$ for (a) $Bi_w = 10$ (case I), $Bi_g = 10$ (case II) and (b) $Bi_w = \infty$ (case I), $Bi_g = \infty$ (case II) as functions of ϕ. Solid lines are used for case I, dotted lines are used for case II.

Appendix A

Confluent hypergeometric functions can be written in terms of Whittaker's function:

$$M_{k,m}(x) = x^{m+(1/2)} \exp\left(-\frac{x}{2}\right)\Phi\left(m-k+\frac{1}{2}, 2m+1; x\right).$$

Asymptotic expressions for $M_{k,m}(x)$ as $x \to \infty$ are given in [8]. First of all,

$$M_{k,m}(4kx) \cong 2^{2/3}\Gamma\Big(2m+1\Big)k^{m+(1/6)}\Big\{\exp\Big[\Big(m-k+\frac{1}{6}\Big)\pi\mathrm{i}\Big] \times Ai\Big[2^{-2/3}(4k)^{2/3}\Big(x-1\Big)\exp\Big(\frac{2\pi\mathrm{i}}{3}\Big)\Big] + \exp\Big[-\Big(m-k+\frac{1}{6}\Big)\pi\mathrm{i}\Big] \times Ai\Big[2^{-2/3}(4k)^{2/3}\Big(x-1\Big)\exp\big(-\frac{2\pi\mathrm{i}}{3}\Big)\Big]\Big\},$$

$$x = 1 + \mathrm{O}\big(k^{-2/3}\big), \tag{A.1}$$

where $Ai(x)$ is Airy's function:

$$Ai(x) = \frac{3^{-2/3}}{\Gamma(2/3)}\left(1+\sum_{n=1}^{\infty}\frac{3^n(1/3)_n}{(3n)!}x^{3n}\right) - \frac{3^{-1/3}}{\Gamma(1/3)}\left(x+\sum_{n=1}^{\infty}\frac{3^n(2/3)_n}{(3n+1)!}x^{3n+1}\right),$$

$$3^n(\alpha+1/3)_n = (3\alpha+1)(3\alpha+4)\cdots(3\alpha+3n-2).$$

Secondly, [8] also shows that

$$M_{k,m}\big(4k\cos^2\rho\big) \cong \sqrt{\frac{2}{\pi}}\Gamma\Big(2m+1\Big)k^{-m}\frac{1}{\sqrt{\tan\rho}} \times \cos\Big[k(\sin(2\rho)-2\rho+\pi) - \Big(m+\frac{\pi}{4}\Big)\Big], \quad -\frac{\pi}{2}<\rho<\frac{\pi}{2}. \tag{A.2}$$

Using (A.1) and (A.2), the following asymptotic expansions can be immediately found:

$$\Phi\left(\frac{1-\lambda}{4},\frac{1}{2};\lambda\phi^2\right) \cong \Big(1-\phi^2\Big)^{-1/4} \times \cos\left(\frac{\lambda}{4}\zeta(\phi)\right)\exp\left(\frac{\lambda\phi^2}{2}\right) \tag{A.3}$$

and

$$\Phi\left(\frac{3-\lambda}{4},\frac{3}{2};\lambda\phi^2\right) \cong \frac{1}{\lambda|\phi|}\Big(1-\phi^2\Big)^{-1/4} \times \sin\left(\frac{\lambda}{4}\zeta(\phi)\right)\exp\left(\frac{\lambda\phi^2}{2}\right), \tag{A.4}$$

where $\zeta(\phi) = 2|\phi|\sqrt{1-\phi^2} - 2\arccos(|\phi|) + \pi$;

$$\Phi\left(\frac{1-\lambda}{4},\frac{1}{2};\lambda\right) \cong \sqrt{\pi}\left(\frac{\lambda}{2}\right)^{-5/6}\frac{3^{-2/3}}{\Gamma(2/3)} \times \sin\left(\frac{1-\lambda}{4}\pi+\frac{\pi}{6}\right)\exp\left(\frac{\lambda}{2}\right) \tag{A.5}$$

and

$$\Phi\left(\frac{3-\lambda}{4},\frac{3}{2};\lambda\right) \cong \sqrt{\pi}\left(\frac{\lambda}{2}\right)^{-5/6}\frac{3^{-2/3}}{\Gamma(2/3)} \times \cos\left(\frac{1-\lambda}{4}\pi+\frac{\pi}{6}\right)\exp\left(\frac{\lambda}{2}\right). \tag{A.6}$$

Asymptotic expressions for $(\Phi(5-\lambda))/4, (3/2); \lambda\xi^2$ will now be derived. If $\xi = \phi$, then we find from Eq. (A.2) that

$$\Phi\left(\frac{5-\lambda}{4},\frac{3}{2};\lambda\phi^2\right) = \big(\lambda-2\big)^{-1/4}\big(\lambda\phi^2\big)^{-3/4} \times \frac{1}{\sqrt{\tan\big(\rho(\lambda,\phi)\big)}} \times \sin\left(\frac{\lambda-2}{4}\zeta_1(\lambda,\phi)\right)\exp\left(\frac{\lambda\phi^2}{2}\right),$$

where $\rho(\lambda,\phi) = \arccos\big(\sqrt{\lambda/(\lambda-2)}|\phi|\big)$ and $\zeta_1(\lambda,\phi) = \sin(2\rho(\lambda,\phi)) - 2\rho(\lambda,\phi) + \pi$. As $\lambda\to\infty$ $\rho(\lambda,\phi)\to\arccos(|\phi|)$, so

$$\zeta_1(\lambda,\phi) = \zeta(\phi) + \frac{2|\phi|}{\lambda\sqrt{1-\phi^2}}\Big(2-2\phi^2\Big) + \mathrm{O}\left(\frac{1}{\lambda^2}\right)$$

and

$$\frac{\lambda-2}{4}\zeta_1(\lambda,\phi) \to \frac{\lambda-2}{4}\left(\zeta(\phi)+\frac{4|\phi|}{\lambda}\sqrt{1-\phi^2}\right) \to \frac{\lambda}{4}\zeta(\phi) - \arcsin(|\phi|).$$

This gives

$$\Phi\left(\frac{5-\lambda}{4},\frac{3}{2};\lambda\phi^2\right) \cong \frac{\big(\phi^2\big)^{-1/2}\big(1-\phi^2\big)^{-1/4}}{\sqrt{\lambda(\lambda-2)}}\sin\left(\frac{\lambda}{4}\zeta(\phi) - \arcsin(|\phi|)\right)\exp\left(\frac{\lambda\phi^2}{2}\right). \tag{A.7}$$

In the same way, it can be shown that

$$\Phi\left(\frac{3-\lambda}{4},\frac{1}{2};\lambda\phi^2\right) \cong \big(1-\phi^2\big)^{-1/4}\cos\left(\frac{\lambda}{4}\zeta(\phi) - \arcsin(|\phi|)\right)\exp\left(\frac{\lambda\phi^2}{2}\right). \tag{A.8}$$

If $\xi = 1$, then we will take the first terms in each of the sums in (A.1):

$$\Phi\left(\frac{5-\lambda}{4},\frac{3}{2};\lambda\right) \cong -\sqrt{\pi}2^{5/6}\lambda^{-5/6}\left[\frac{3^{-2/3}}{\Gamma(2/3)}\sin\left(\frac{1-\lambda}{4}\pi+\frac{\pi}{6}\right) - \frac{3^{-1/3}}{\Gamma(1/3)}\left(\frac{2}{\lambda-2}\right)^{1/3}\sin\left(\frac{1-\lambda}{4}\pi+\frac{5\pi}{6}\right)\right]\exp\left(\frac{\lambda}{2}\right) \tag{A.9}$$

and

$$\Phi\left(\frac{3-\lambda}{4},\frac{1}{2};\lambda\right) \cong \sqrt{\pi}2^{5/6}\lambda^{1/6}\left[\frac{3^{-2/3}}{\Gamma(2/3)}\cos\left(\frac{1-\lambda}{4}\pi+\frac{\pi}{6}\right) - \frac{3^{-1/3}}{\Gamma(1/3)}\left(\frac{2}{\lambda-2}\right)^{1/3}\cos\left(\frac{1-\lambda}{4}\pi+\frac{5\pi}{6}\right)\right]\exp\left(\frac{\lambda}{2}\right). \tag{A.10}$$

Appendix B

Let $f(x)$ be a solution to the differential equation

$$\frac{d}{dx}\left(p(x)\frac{df}{dx}\right) + \left(\lambda^2 q(x) + r(x)\right)f = 0,$$

where $p(x)$, $q(x)$ and $r(x)$ are continuously differentiable functions and λ is a parameter. Then the following equality is valid:

$$\int q(x)f^2(x)\,dx = \frac{p(x)}{2\lambda}\left(\frac{\partial f}{\partial \lambda}\frac{\partial f}{\partial x} - f\frac{\partial^2 f}{\partial\lambda\partial x}\right) + \text{const}. \tag{B.1}$$

Indeed, differentiation of both sides of (B.1) by x gives

$$\begin{aligned} q(x)f^2(x) &= \frac{p(x)}{2\lambda}\left(\frac{\partial f}{\partial\lambda}\frac{\partial^2 f}{\partial x^2} - f\frac{\partial^3 f}{\partial\lambda\partial x^2}\right) + \frac{1}{2\lambda}\frac{dp}{dx}\left(\frac{\partial f}{\partial\lambda}\frac{\partial f}{\partial x} - f\frac{\partial^2 f}{\partial\lambda\partial x}\right) \\ &= \frac{1}{2\lambda}\frac{\partial f}{\partial\lambda}\left(p\frac{\partial^2 f}{\partial x^2} + \frac{dp}{dx}\frac{\partial f}{\partial x}\right) - \frac{f}{2\lambda}\frac{\partial}{\partial\lambda}\left(p\frac{\partial^2 f}{\partial x^2} + \frac{dp}{dx}\frac{\partial f}{\partial x}\right) \\ &= -\frac{1}{2\lambda}\frac{\partial f}{\partial\lambda}(\lambda^2 q + r)f + \frac{1}{2\lambda}f\frac{\partial}{\partial\lambda}\left((\lambda^2 q + r)f\right) = q(x)f^2(x). \end{aligned}$$

In our case, the function $\Psi(\xi)$ (Eq. (13)) is clearly infinitely differentiable by both ξ and λ and therefore satisfies (B.1), where $p(x) \equiv 1$, $q(x) = 1 - x^2$ and $r(x) \equiv 0$.

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